Euler-Lagrange equations
Monday, June 29th 2015

Problem 1 The functional that describes the length of a $\mathcal{C}^1$ curve parametrized by $x(t),\ t \in [a, b]$, is

$$J(x(t)) = \int_a^b \sqrt{1 + (\dot{x}(t))^2} dt.$$

Use the Euler-Lagrange equations to find the shortest path between two points in the plane.

Problem 2 Consider a general cost functional

$$J(x(t)) = \int_a^b L(t, x(t), \dot{x}(t)) dt$$

defined on the set of smooth curves $x : [a, b] \to \mathbb{R}$ such that $x(a) = x_0$ is fixed and $x(b)$ is free. Derive necessary optimality conditions. (Hint: you should find that both the Euler-Lagrange equations and an additional condition are satisfied).

Problem 3 Consider a general cost functional

$$J(x(t)) = \int_a^{t_f} L(t, x(t), \dot{x}(t)) dt$$

defined on the set of smooth curves $x : [a, t_f] \to \mathbb{R}$ such that $x(a) = x_0$ is fixed, t_f is unspecified and $x(t_f) = \phi(t_f)$ for some $\mathcal{C}^1$ function $\phi : \mathbb{R} \to \mathbb{R}$. Derive necessary optimality conditions. (Hint: you should find that both the Euler-Lagrange equations and an additional condition are satisfied)